# Complex sine-Gordon-2: a new algorithm for multivortex solutions on the plane

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(Dated: February 8, 2008)

## Abstract

We present a new vorticity-raising transformation for the second integrable complexification of the sine-Gordon equation on the plane. The new transformation is a product of four Schlesinger maps of the Painlevé-V to itself, and allows a more efficient construction of the $n$-vortex solution than the previously reported transformation comprising a product of $2n$ maps.

**1.** The complex sine-Gordon equation, also known as the Lund-Regge model, was introduced in 1970s in several field-theoretic contexts [1, 2, 3, 4]. In $(2+0)$-dimensional space, the equation assumes the form

$$\nabla^2\psi + \frac{(\nabla\psi)^2\,\overline{\psi}}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0. \tag{1}$$

(Here and below, $\nabla = \mathbf{i}\partial_x + \mathbf{j}\partial_y$.) We will be referring to eq.(1) as the complex sine-Gordon-1, in order to distinguish it from another integrable complexification of the sine-Gordon theory, the so-called complex sine-Gordon-2:

$$\nabla^2\psi + \frac{(\nabla\psi)^2\overline{\psi}}{2-|\psi|^2} + \tfrac{1}{2}\psi(1-|\psi|^2)(2-|\psi|^2) = 0. \tag{2}$$

This model has also been known since the late 1970s, yet in $(1+1)$-dimensional space [5, 6]. The names stem from the fact that if we assume that $\psi$ is real and substitute $\psi = \sin(\alpha/2)$ in (1) and $\psi = \sqrt{2}\sin(\alpha/4)$ in (2), both systems reduce to the conventional, real, sine-Gordon equation $\nabla^2\alpha + \sin\alpha = 0$. In the physics literature, it is common to define the two models by their action functionals:

$$E_{SG1} = \int \left\{ \frac{|\nabla\psi|^2}{1-|\psi|^2} + (1-|\psi|^2) \right\} \mathrm{d}^2x,$$

and

$$E_{SG2} = \int \left\{ \frac{|\nabla\psi|^2}{1-\frac{1}{2}|\psi|^2} + \tfrac{1}{2}(1-|\psi|^2)^2 \right\} \mathrm{d}^2x,$$

respectively.

Recently there has been an upsurge of interest in the complex sine-Gordon equations, motivated by the fact that they define integrable perturbations of conformal field theories [7, 8, 9, 10]. There is, however, yet another reason for considering these systems more closely; out of all vortex-bearing equations for one complex field, the complex sine-Gordon-1 and -2 are the only equations whose vortex (and multivortex) solutions are available in explicit analytic form [11, 12]. Consequently, they provide a unique source of insight into general properties of topological solitons on the plane. The latter can be of value for a whole range of models like the Gross-Pitaevski and easy-plane ferromagnet equations, where vortices are only available numerically.

The (coaxial) multivortices of the complex sine-Gordon-2 have been obtained via the Schlesinger transformations of the fifth Painlevé equation [11]. The procedure is cumbersome: even if the $(n-1)$-vortex is already available, the construction of the $n$-vortex solution

requires applying the Schlesinger transformation $2n$ times anew. On the contrary, there is an efficient recursive procedure for the complex sine-Gordon-1, allowing a one-step construction of its solution with vorticity $n$ provided the $(n-1)$-vortex solution is known [11, 12]. The purpose of the present note is to formulate a similar recursive procedure for the complex sine-Gordon-2.

**2.** The coaxial $n$-vortex configuration has the form $\psi(r,\theta) = Q_n^{1/2}(r)e^{in\theta}$. Substituting this Ansatz into (2) yields an equation for the radial "amplitude" $Q_n$ which we write as

$$\frac{d^2Q_n}{dr^2} + \frac{1}{r}\frac{dQ_n}{dr} + \frac{1-Q_n}{Q_n(Q_n-2)}\left(\frac{dQ_n}{dr}\right)^2 + Q_n(1-Q_n)(2-Q_n) + \frac{(a^2-b^2)Q_n}{r^2(2-Q_n)} + \frac{4a^2(1-Q_n)}{r^2Q_n(2-Q_n)} + \frac{\gamma Q_n(2-Q_n)}{2r} = 0, \quad (3)$$

with $a = \gamma = 0$ and $b = -2n$. The last two terms in (3) being equal to zero, this form may appear to be somewhat artificial. However, there is an advantage in considering eq.(3) with general $a$, $b$, and $\gamma$; namely, the availability of transformations connecting solutions with different sets of parameters. Indeed, the change of variables [11]

$$Q_n = \frac{2}{1-W} \quad (4)$$

brings eq.(3) to the fifth Painlevé equation,

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} - \frac{3W-1}{2W(W-1)}\left(\frac{dW}{dr}\right)^2 = \frac{(W-1)^2}{r^2}\left(\alpha W + \frac{\beta}{W}\right) + \frac{\gamma W}{r} + \delta\frac{W(W+1)}{W-1}, \quad (5)$$

with $\alpha = \frac{1}{2}a^2$, $\beta = -\frac{1}{2}b^2$ and $\delta = 2$. The Painlevé-V is covariant under the Schlesinger transformation [13, 14] which takes a solution $W$ with the parameter values $a, b, \gamma$ and $\delta$ to a solution $\hat{W}$ of the same equation with parameter values

$$\hat{a} = \tfrac{1}{2}(a+b-1-\gamma/c), \quad \hat{b} = \tfrac{1}{2}(a+b-1+\gamma/c), \quad \hat{\gamma} = c(b-a),$$

and $\hat{\delta} = \delta$. Here $c$ is one of the two values with $c^2 = -2\delta$; in our case we can set, without loss of generality, $c = 2i$. Written in terms of $Q$ and $\hat{Q} = 2(1-\hat{W})^{-1}$, the direct and inverse Schlesinger transformations have the form

$$\hat{Q} = 1 - \frac{i}{Q(Q-2)}\left[\frac{dQ}{dr} + \frac{Q(a+b)-2a}{r}\right], \quad (6a)$$

$$Q = 1 + \frac{i}{\hat{Q}(\hat{Q}-2)}\left[\frac{d\hat{Q}}{dr} - \frac{(\hat{Q}-1)(a+b-1)}{r} + \frac{i\gamma}{2r}\right]. \quad (6b)$$

If $Q^{(0)} = Q_n$ is a solution of eq.(3) with parameters $a^{(0)} = \gamma^{(0)} = 0$, $b^{(0)} = -2n$, then applying transformation (6a) we obtain a solution $\hat{Q} = Q^{(1)}$ with parameters

$$\begin{aligned} a^{(1)} &= \tfrac{1}{2}(a^{(0)} + b^{(0)} - 1 - \gamma^{(0)}/c) = -n - \tfrac{1}{2}, \\ b^{(1)} &= \tfrac{1}{2}(a^{(0)} + b^{(0)} - 1 + \gamma^{(0)}/c) = -n - \tfrac{1}{2}, \\ \gamma^{(1)} &= 2i(b^{(0)} - a^{(0)}) = -4in. \end{aligned}$$

Note that since $a^{(1)}$ and $\gamma^{(1)}$ are not zero and $b^{(1)}$ is not a negative even integer, $Q^{(1)}$ does not represent the amplitude of any multivortex. Using (6a) again, this time with $Q = Q^{(1)}$ and $\hat{Q} = Q^{(2)}$, yields a solution $Q^{(2)}$ with parameters

$$\begin{aligned} a^{(2)} &= \tfrac{1}{2}(a^{(1)} + b^{(1)} - 1 - \gamma^{(1)}/c) = -1, \\ b^{(2)} &= \tfrac{1}{2}(a^{(1)} + b^{(1)} - 1 + \gamma^{(1)}/c) = -2n - 1, \\ \gamma^{(2)} &= 2i(b^{(1)} - a^{(1)}) = 0. \end{aligned}$$

Thus, the effect of the product transformation $Q^{(0)} \to Q^{(2)}$ is to reduce both $a$ and $b$ by one. Although $\gamma$ is now zero, $Q^{(2)}$ is still not a multivortex (since $a^{(2)}$ is nonzero). The crucial observation now is that (3) depends only on the square of $a$; hence $Q^{(2)}$ is a solution of eq.(3) not only for $a = -1$, $b = -2n - 1$, but also for $\tilde{a} = +1$, $\tilde{b} = -2n - 1$. Repeating the above two transformations will decrease both $\tilde{a}$ and $\tilde{b}$ by one more, yielding $a^{(4)} = 0$, $b^{(4)} = -2n - 2$. The corresponding solution $Q^{(4)}$ will therefore be $Q_{n+1}$, the multivortex with vorticity $n + 1$. Thus, the transformation $Q^{(0)} \to Q^{(4)}$, comprising a product of four Schlesinger maps, is nothing but a vorticity-raising transformation $Q_n \to Q_{n+1}$.

Starting with the trivial "vortex" $Q_0 = 1$ and applying the transformation $Q_n \to Q_{n+1}$ recursively, one can construct multivortices of any desired vorticity. It follows from the form of the transformation that all $Q_n$'s will be rational functions of $r$. The one-, two- and three-vortex amplitudes are:

$$Q_1 = \frac{r^2}{r^2 + 4},$$

$$Q_2 = \frac{r^4(r^2 + 24)^2}{r^8 + 64r^6 + 1152r^4 + 9216r^2 + 36864},$$

and

$$Q_3 = \frac{r^6(r^6 + 144r^4 + 5760r^2 + 92160)^2}{D_3},$$

where

$$D_3 = r^{18} + 324r^{16} + 41472r^{14} + 2820096r^{12} + 114130944r^{10} + 2919628800r^8$$
$$+50960793600r^6 + 61152952300r^4 + 4892236185600r^2 + 19568944742400.$$

These solutions coincide with those constructed previously in [11].

**3.** As another application of the new vorticity-raising transformation, we construct one more class of vortex-like solutions. Unlike the solutions discussed above, these solutions of the complex sine-Gordon-2 decay to their asymptotic value in an *oscillatory* fashion. (The asymptotic value is now $\sqrt{2}$ not 1.)

We start with the radially symmetric solution [11] of the complex sine-Gordon-1 with vorticity $n = 2$:

$$\Phi_2 = -\frac{I_0 I_2 - I_1^2}{I_0^2 - I_1^2}.$$

Here, $I_m = I_m(r)$ is the modified Bessel function of order $m$. The change of variables

$$\Phi_2 = \frac{1+W}{1-W}$$

transforms it to a solution

$$W_2(r) = \frac{I_0 I_1 - r(I_0^2 - I_1^2)}{I_0 I_1} \tag{7}$$

of the Painlevé-V, eq.(5) with parameter values

$$\alpha = \tfrac{1}{2}, \quad \beta = -\tfrac{1}{2}, \quad \gamma = 0, \quad \delta = -2.$$

(In (7), we simplified, using the standard recurrence relation $I_0 - I_2 = 2I_1/r$.) Next, making the replacement $r \to ir$ amounts to replacing $\gamma \to i\gamma$, $\delta \to -\delta$ in (5); hence

$$\tilde{W}_2(r) = W_2(ir) = \frac{J_0 J_1 - r(J_0^2 + J_1^2)}{J_0 J_1}$$

is a solution to the Painlevé-V with

$$\alpha = \tfrac{1}{2}, \quad \beta = -\tfrac{1}{2}, \quad \gamma = 0, \quad \delta = 2. \tag{8}$$

Here, $J_m = J_m(r)$ is the ordinary Bessel function of order $m$.

For the parameter values in (8), the change of variables (4) brings the Painlevé-V to equation (3) with $a = 1$ and $b = -1$. Thus we can use (4) followed by two applications of

(6a) to reduce both $a$ and $b$ by one (as described in the previous section). The result is a solution to the complex sine-Gordon-2 with $n = 1$:

$$\tilde{Q}_1 = \frac{2\left[r(J_0^2 + J_1^2) - J_0 J_1\right]^2}{r^2(J_0^2 + J_1^2)^2 + J_0^4}. \tag{9}$$

Applying the vorticity-raising transformation to $\tilde{Q}_1$ gives the formula for $\tilde{Q}_2$, the 2-vortex amplitude:

$$\tilde{Q}_2 = 2\frac{P_2^2}{D_2}, \tag{10}$$

where

$$P_2 = 4r^3 J_0^4 + 8r^3 J_0^2 J_1^2 + 4r^3 J_1^4 - 8r^2 J_0^3 J_1 - 8r^2 J_0 J_1^3 + 3r J_0^4 \\ + 2r J_0^2 J_1^2 - 5r J_1^4 - 6J_0^3 J_1 + 2J_0 J_1^3,$$

and

$$D_2 = 416r^4 J_1^2 J_0^6 - 128r^5 J_0 J_1^7 + 736r^4 J_1^6 J_0^2 - 384r^5 J_0^3 J_1^5 + 96r^6 J_1^4 J_0^4 - 12r^2 J_1^2 J_0^6 \\ - 96r J_0^3 J_1^5 - 320r^3 J_0 J_1^7 - 32r J_0^5 J_1^3 + 318r^2 J_1^4 J_0^4 + 64r^6 J_1^2 J_0^6 - 128r^5 J_0^7 J_1 \\ - 832r^3 J_0^3 J_1^5 + 260r^2 J_1^6 J_0^2 - 448r^3 J_0^5 J_1^3 - 384r^5 J_0^5 J_1^3 + 1008r^4 J_1^4 J_0^4 + 64r^3 J_0^7 J_1 \\ + r^2 J_1^8 + 152r^4 J_1^8 + 9r^2 J_0^8 + 16r^6 J_1^8 + 16r^6 J_0^8 + 16J_1^4 J_0^4 - 8r^4 J_0^8 + 64r^6 J_1^6 J_0^2.$$

(The actual moduli $\tilde{\Phi}_1 = \sqrt{\tilde{Q}_1(r)}$ and $\tilde{\Phi}_2 = \sqrt{\tilde{Q}_2(r)}$ are shown in figure 1.) Proceeding recursively we can construct $\tilde{Q}_n$ with arbitrary $n$. For $n \geq 3$, the formulas become intractable, so we only produce the asymptotic behaviours, as $r \to \infty$:

$$\tilde{Q}_n \to 2 + (-1)^{n+1}\frac{2n}{r}\cos(2r) + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{11}$$

Eq.(11) can be easily proved by induction.

This project was supported by the NRF of South Africa under grant 2053723, by the Johnson Bequst Fund and the URC of the University of Cape Town.

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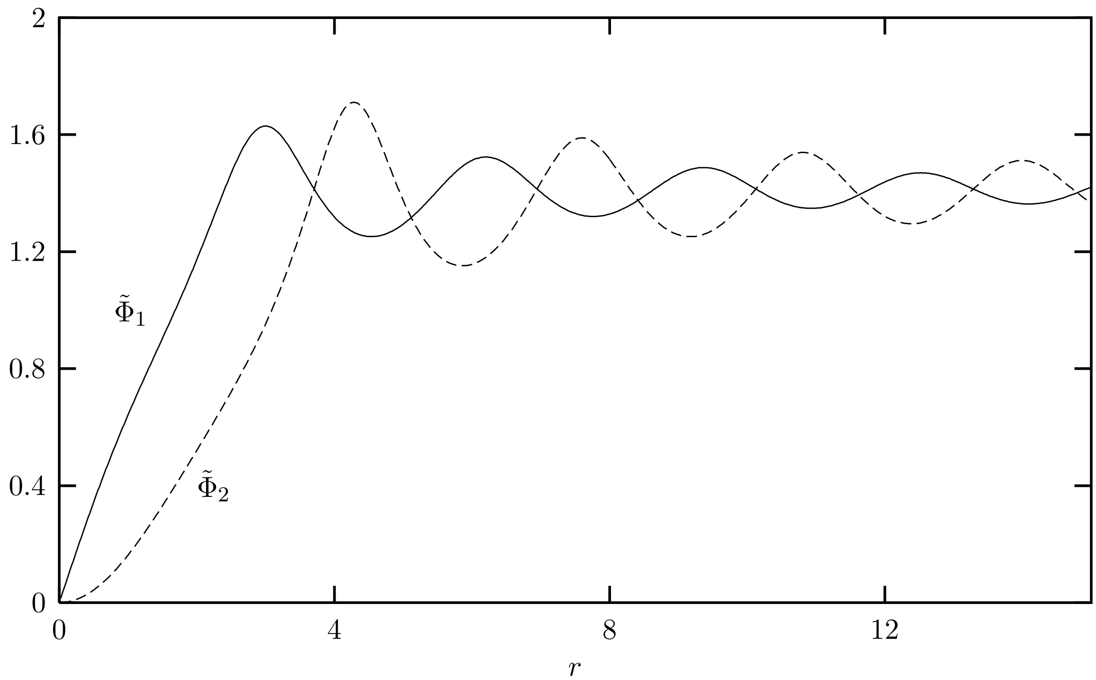


FIG. 1: The moduli of the 1 and 2-vortex solutions (9) and (10). (As $r \to \infty$, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ tend to $\sqrt{2}$.)


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